Proof for Tripartite Graphs

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Let Tripartite graphs be graphs with three sets of points, each set located on one side of a triangle. Similar to bipartite graphs, the function ☾2,1,1) would be a graph with one side having two points and the other two sides having one point each, and the result of the function being equal to the number of intersections between lines when each point in each set of points is mutually connected to each point in both of the other sets of points with a line (See figure A). For ☾2,1,1), the result would be 1 because there is only one intersection.

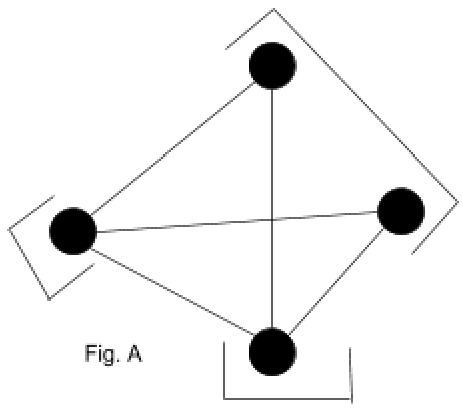


Fig. A

Let m☆n be the number of intersections in a bipartite graph with m points on one side and n points on the other. We have shown that m☆n = △(m-1)△(n-1) where △ is the triangular number function (See Appendix A).

For tripartite graphs, the number of dots on each side relates to the number of intersections by the equation:

☾m, n, q) =
$\frac{1}{4}[m^2n^2+m^2q^2+n^2q^2-m^2n-n^2m-m^2q-q^2m-n^2q-q^2n+mn+mq+nq+2m^2nq+2n^2mq+2q^2mn-6mnq]$, where m, n, and q are letters representing the number of points on each side.

Consider the tripartite graph of m,n,q as the combination of three bipartite graphs: m with (n+q) (the m-pair), n with (q+m) (the n-pair), and q with (m+n) (the q-pair). However, we must take into account the number of intersections created by the complex interplay of these bipartite graphs. We have overcounted! The bipartite graphs of m with (n+q) and n with (q+m) both share the bipartite graph of m with n, causing its intersections to be counted twice. (See Figures B, C and D; overcounted intersection is circled.)

Figure B:

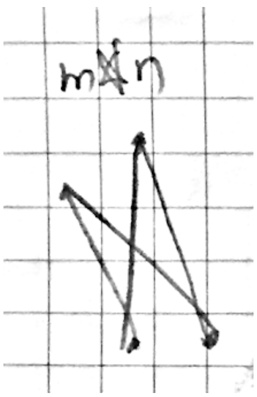


Figure C:

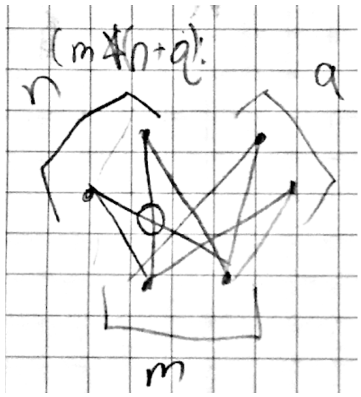
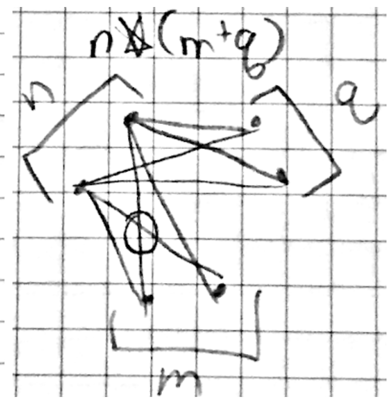


Figure D:



☾ m,n,q) is therefore found by summing the star of the m-pair, the n-pair, and the q-pair, and then subtracting the overcounted intersections. It can be written as m☆(n+q)+n☆(m+q)+q☆(m+n)-m☆n-m☆q-n☆q. Using our previous bipartite proof, this can be rewritten as △(m-1)△(n+q-1)+△(n-1)△(m+q-1)+△(q-1)△(m+n-1)-△(m-1)△(n-1)-△(m-1)△(q-1)-△(n-1)△(q-1), OR ☾ m, n, q) =

$\frac{1}{4}[m^2n^2+m^2q^2+n^2q^2-m^2n-n^2m-q^2m-m^2q-n^2q-q^2n+mn+mq+nq+2m^2nq+2n^2mq+2q^2mn-6mnq]$

Appendix A
(A proof by induction)

For a planar Euclidean bipartite graph, let m be the number of points in the upper row, and let n be the number of points in the lower row. Let m☆n be equal to the number of intersections between two edges when every point in the upper row is connected by an edge to every point in the lower row. We will show that m☆n=△(m-1)△(n-1)=[nm(n-1)(m-1)]/4.

Base Case:

We will show that this formula works for the base case 1☆1. There is one dot on the top and one dot on the bottom, and a line connects them. Obviously a straight line cannot intersect itself making zero intersections, and so 1☆1 is equal to zero. △(1-1), or the 0th triangular number, is also obviously 0. Substituting this into the second half of the equation and simplifying, we get 0*0. Therefore, the second half of the equation is 0 as well. Thus, we have shown that our formula holds for the base case of 1☆1; yay!, we did it.

Inductive Step:

Assume that m☆n does indeed equal △(m-1)△(n-1) for some natural numbers m,n. We will show that this formula works for m☆(n+1). You should also know that ☆ is commutative because the top and the bottom of a bipartite graph can be switched. This means that do not have to prove that m can be increment separately.

The upper row of vertices of the bipartite graph will be called A and the lower row B. Each vertex in a row will be called by the name of its row sub its distance from the left starting with 1. (ie A_1, A_2, etc. and B_1, B_2 , etc.). Each edge is called by its two endpoints (ie A_1B_2).

Recall that △(x) = (x*(x+1))/2

Suppose that when adding a new point to B, you always add it on the right. Let's call the new point B_n. The edge B_nA_1 will cross all of the pre-existing edges except those with an endpoint of A_1. Therefore, the number of new intersections will be equal to n(m-1). Then each following edge (B_nA_2 B_nA_3) will cross n fewer edges than the previous added edge.

In other words,we are adding n(m-1)+n(m-2)+...+n(2)+n(1).a
Factoring out the n gives us the following number of intersections:
△(m-1)*△(n-1) + n*△(m-1) =
△(m-1)*△(n-1)+n) =
△(m-1)*△(n) This is equal to m☆(n+1) We are now done with the inductive step.

Because we have established a base case and we can increment from any m and n that follow the formula to the next case, we have now proven that the formula will hold for all natural numbers n and m.

Try to find an n,m where it doesn't work. You won't find one! Boo-yah!

Quod Erat Demonstrandum!