Euler Trails: Leader Instructions

Once upon a time there was a city called Königsburg that had a branching river, an island, and some bridges. The people of Königsburg decided to try to find a way to design a walking tour that would cross each bridge exactly once and then return to the starting place, but no one could do it. In this activity, students try several puzzles of this type and solve the mystery of the bridges of Königsburg. They determine when it is possible to cross each bridge in a riverwalk exactly once.

Levels 1st through 6th grades

Topics Graph Theory, Even and Odd Numbers, Problem Solving

Goals

- Students will learn what *graph theory* is.
- Students will learn what *vertices* and *edges* are.
- Students will learn the definition of an *Euler trail.*
- Students will learn that the *degree* of a vertex is the number of edges coming into or out of it.
- Students will determine whether the degree of each vertex is even or odd.
- Students will learn how to tell when a graph has an Euler trail and when it does not.

Preparation Time 30 minutes if building or drawing maps; none otherwise

Activity Time 35 to 60 minutes

Materials and Preparation

- Sidewalk chalk and pavement OR large paper cut-outs arranged on the floor to represent rivers and bridges OR handouts and pencils for working with maps.
- Each group needs 20 tokens to mark paths unless you are working on paper.
- If you are working with paper, each student needs a handout.
- Each group needs chalk OR small squares of paper with letters A, B, C, and D unless you are working on paper. (Do not use these labels at the beginning.)
- If using sidewalk chalk or paper cut-outs, draw or build at least one map for every group of two to four students.
- Each student needs several pieces of scrap paper.
- Each student needs a pencil.

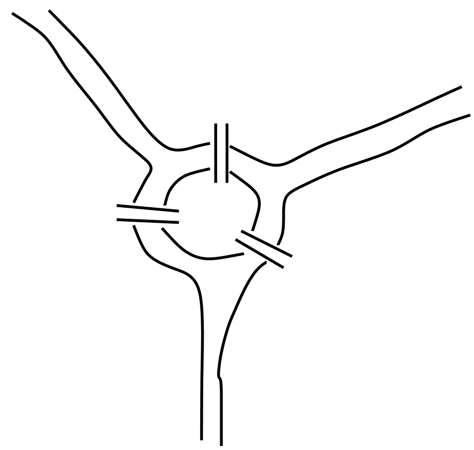
Primary Source University of Alabama Center for Teaching and Learning: Math 103. "Euler Paths and Circuits." http://www.ctl.ua.edu/math103/

Background

Allow 5 minutes for the background discussion. Tell the students that they will be working in an area of mathematics called *graph theory.* Graph theory is the study of how different places are connected to one another. Graph theorists study links between websites on the internet, find efficient methods for shipping things to stores, and plan airline flights.

There was once a city in East Prussia called Königsburg. (Today, Königsberg is called Kaliningrad and it is located in Russia.) In the 1750s, Königsburg had seven bridges as shown on the handout. Many people tried to design a walking route that would cross each bridge exactly once. No one could solve the problem.

When Leonhard Euler (pronounced "oiler") heard about the challenge, he considered lots of arrangments of rivers and bridges. He realized that it is sometimes impossible to walk on each bridge exactly once. For example, consider the map shown below. This map shows a central island with three bridges leading out to three pieces of land. (The pieces of land are separated by rivers and we are assuming that the three rivers keep going out for a long way so that we cannot walk around them.) Why is it impossible to walk on each bridge exactly once no matter where we start?



An Euler trail is a path that crosses each bridge exactly once. Demonstrate how to look for an Euler trail. If the students are working with sidewalk chalk drawings or paper cut-outs, they should choose a piece of land to start on. They should choose which bridge they want to cross first and place a token on that bridge so they know they used it. They should then step across the river to the other side of the bridge. They should repeat this process until they either realize they are stuck or they have placed a token on every bridge. Emphasize that they can only cross a bridge that touches the piece of land they are currently standing on. Remind them that we assume that the rivers keep going at the edges of the map so that they cannot just walk around the ends of the rivers.

If students are using a pencil, they can draw their paths on the handout.

Activity Instructions

Allow 10 to 20 minutes for students to try the different maps and decide which ones have Euler trails. Students should work in teams of 2 to 4 people and they should try as many different maps as possible. They may have to try some maps several times starting from different pieces of land in order to find an Euler trail. However, some of the maps might not contain an Euler trail, so they should not stay with a map too long if it is not working. As the students are working on the maps, ask them to think about why some riverwalks have an Euler trail and why others do not. What is the difference between them? Ask them whether it seems to matter which piece of land they start on.

Drawing Graphs

Allow 10 minutes to show students how to draw graphs based on the maps.

When everyone has had a chance to try several maps, gather everyone together. For each map, ask whether anyone found an Euler trail. Did they notice anything different about the maps that have Euler trails and those that do not? Did it matter where they started or ended?

Demonstrate the method Leonhard Euler used to represent the land and bridges in a riverwalk. To make a *graph*, begin by placing letter labels on each piece of land (see the Königsberg/Kaliningrad maps for examples of graphs corresponding to maps). On a piece of scrap paper, write the four letters in roughly the same configuration as they are on the map. Draw a dot near each letter. For each bridge in the map, notice which two pieces of land are connected by the bridge. Draw a path between the two corresponding dots on the graph. The path does not need to be a straight line. If two bridges connect the same piece of land, then there will be two paths between those dots. It is best to make these two paths bend in different directions so that you can see both paths clearly.

We call each dot on the graph a *vertex*, and each path is called an *edge*. The number of edges emerging from each vertex is called the *degree* of the vertex. For example, consider the first map on the handout, showing Königsberg with hypothetical bridges. The degree of vertex A is 2, the degree of vertex B is 2, the degree of vertex C is 4, and the degree of vertex D is 2. Write the degree next to each vertex.

Ask each group of students to draw a graph of one of the maps and have them label the vertices with the appropriate degree numbers.

Conclusions

Allow 10 minutes for the concluding discussion.

Once the graphs are drawn and the degrees are determined, gather the students together again. As a group, list the degrees of all vertices for each map that had an Euler trail. Make another list for maps where no one found an Euler trail.

Students may notice that graphs with vertices of even degree always have an Euler trail. Graphs that have more than two vertices of odd degree are actually impossible to solve. If there are exactly two vertices of odd degree, then there is an Euler trail. However, the trail must start at one of the odd vertices and it must end at the other odd vertex. (See the section below for a discussion of why vertices of odd degree cause difficulties.)

If you have a younger group, you may wish to review the definitions of even and odd numbers or ask students to list examples of each type. It can be helpful to have groups of students stand up to represent even and odd numbers. If there are an even number of people in the group, then it is possible to form a double line where everyone has a partner. If there are an odd number of people, then someone will be left without a partner.

Have the students re-visit the maps to confirm that odd vertices cause problems. Students should mark each piece of land with a green or red token depending on whether there are an even or odd number of bridges touching it. If there are more than two red tokens (more than two vertices of odd degree), then it will be impossible to find an Euler trail. If there are exactly two red tokens, then the Euler trail must start at one of them and end at the other. If the tokens are all green, then you can start anywhere and still find an Euler trail.

At the end of the discussion, consider the map of Königsburg in 1750 again. Why did people have such a hard time finding an Euler trail? Why doesn't that map have an Euler trail?

Taking it Further

Students might enjoy building or drawing their own maps with rivers and bridges. They can count the number of bridges touching each piece of land to determine whether or not there is an Euler trail. If there is one, they can try to find it. Students may like to challenge each other to find Euler trails.

Older students might enjoy thinking about strategies for finding an Euler trail. If they were going to program a robot to find an Euler trail, what rules should it follow in order to succeed? (Here is one method. Start at an odd vertex if there is one. Otherwise, start anywhere. To find the next step in the trail, we randomly choose one of the edges touching the vertex we are at. If removing that edge would break the collection of unused edges into two disconnected pieces, then we do not take that edge and we try another edge instead. If removing the chosen edge leaves a connected collection behind, then we go ahead and add that edge to our trail. We continue this way until we have used all of the edges.)

Why Odd Degrees Cause Difficulties

It is not possible to have an Euler trail through a vertex of odd degree if you do not either start or end at that vertex. To see this, draw a single vertex with an odd number of edges sticking out of it. Let's assume that we are not going to start or end on this vertex. That means that we will have to use one of the edges to come in to the vertex. We will use another edge to leave the vertex. We will use another edge to come in to the vertex. We will use another edge to leave the vertex. If we continue this way on a vertex of odd degree, we see that the last edge must be used to come in to the vertex. This means that it is not possible for us to leave that vertex again and we are stuck at that vertex. However, we assumed that we were not going to start or end the Euler trail at this vertex. This argument shows why a vertex of odd degree must either be the starting point for the Euler trail or the ending point for the trail.

If a graph has more than two vertices of odd degree, there is no way to find a trail. This is because a trail has one beginning and one end and so only two bad odd-degree vertices can be accommodated.

Students sometimes ask whether it is possible to find an Euler trail for a graph that has only one vertex of odd degree. The answer to this question is that it is actually impossible for a graph to have only one odd vertex. In fact, it is impossible for a graph to contain three odd vertices, five odd vertices, or any other odd number of odd vertices. Why is this true? Suppose that we added up the degrees of all the vertices in a graph. We could think of that total number as just counting both ends of each edge in the graph. Each edge has two ends, and so the sum of all the degrees must be an even number.

When we add an even number to an even number, the result is always even. (If we represent each even number as a group of people who are lined up in pairs, everyone still has a partner if we push the two groups together.) If we add two odd numbers together, we get an even number because the person in the first group who had no partner can be paired with the person in the other group who had no partner. If we add an even and an odd number together, then the result will be odd.

Suppose that a graph had only one vertex of odd degree. To find the sum of all the degrees we could first add up all the even numbers first. The result would be even. Then we could add the odd degree to this total. The result would be odd. However, we know that the sum of all the degrees must be even so it would be impossible for only one vertex to be odd.

Similarly, if we had an odd number of vertices of odd degree we would end up with an odd number. (Each pair of odd vertices adds up to an even number, but there is one odd vertex left over without a partner.)