Catz, Ratz, and Batz Game: Lesson Plan

Students learn to play a dice game called Catz, Ratz, and Batz. Analysis of the game reveals a quadratic function that governs the maximum possible score on each turn. Students also consider questions of expected value corresponding to various choices in the game to determine what is the best option.

Levels 7th through 12th grade

Topics Multiplication facts, Quadratic functions, Modeling, Probabilty, Expected value

Goals

- Students will explore a scenario that can be modeled by a quadratic function.
- Students will compute expected values to discover the best strategy for certain situations in a game.

Pre-requisite Knowledge Multiplication facts, Familiarity with quadratic functions, Familiarity with expected value and probability concepts.

Preparation Time 5 minutes.

Activity Time 60 to 120 minutes.

Materials and Preparation

- Notebook paper
- Graph paper
- Pencils
- Catz, Ratz, and Batz games for groups to play.

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Sources

Catz, Ratz, and Batz. Created by Maureen Hiron. Published by Playroom Entertainment in 2003.

Helpful Hints for Activity Leaders

What is the Highest Possible Score?

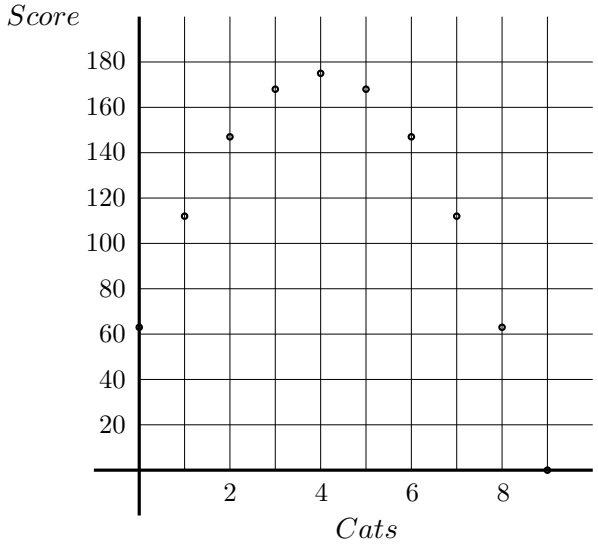
To obtain the highest score in the Catz, Ratz, and Batz game, it would be best to have some combination of Cats and 7s on the nine dice. The table below shows all of these combinations:

Cats	7s	Total Score
0	9	63
1	8	112
2	7	147
3	6	168
4	5	175
5	4	168
6	3	147
7	2	112
8	1	63
9	0	0

From the table, it is clear that the best score occurs when there are four Cats and five 7s. In this case, the base score would be $7 \cdot 5 = 35$. The four cats would add four additional copies of this score to the base score – in other words, they effectively multiply the score by 5. The total score is then $35 \cdot 5 = 175$.

Graphing the Points

If we plot the points from the table, we see that the values lie along a curve that looks as though it might be a parabola.



Formula for the Total Score as a function of the Number of Cats

Let C stand for the number of Cats. Then $9 - C$ stands for the number of 7s. The base score will be obtained by multiplying the number of 7s by seven. So the base score is $7(9 - C)$. This base score will be multiplied by one more than the number of Cats to obtain the total score. Therefore the total score is given by the following formula:

$$\begin{aligned} \text{Total Score} &= 7(9 - C)(C + 1) \\ &= -7C^2 + 56C + 63 \end{aligned}$$

We can see that this is a quadratic function which confirms that the points we plotted in the last section do indeed lie along a parabola. The vertex of this parabola is at $(4, 175)$ which corresponds to the maximum possible score for a turn of Catz, Ratz, and Batz.

The reason that this function turns out to be a quadratic function is that the base score and the multiplier are competing for how many dice are allocated for each purpose. The largest product is obtained when these two values are identical. When one is larger and the other is smaller, the result is not as large. This is reminiscent of problems where the area of a rectangle is maximized while the sum of the length and width is fixed.

How Much do Rats Lower Your Potential Score?

Suppose that one of the dice shows a Rat, but we are allowed to select the other dice so that the score is as large as possible. At least one Cat must be allocated to cancel out the Rat. The remaining seven dice can be chosen to be whichever combination of Cats and 7s gives the largest score. The formula for the total score would then be If C stands for the number of cats on the remaining seven dice (not counting the one needed to cancel the rat), then the formula for the total score is:

$$\begin{aligned} \text{Total Score} &= 7(7 - C)(C + 1) \\ &= -7C^2 + 42C + 49 \end{aligned}$$

This parabola has its maximum at the vertex at $(3, 112)$

In general, each additional mandated Rat reduces the number of dice that count towards the total score by two. Thus, if R is equal to the number of mandated Rats, the Total Score is given by

$$\begin{aligned} \text{Total Score} &= 7(9 - 2R - C)(C + 1) \\ &= -7C^2 + (56 - 14R)C + (63 - 14R) \end{aligned}$$

The vertex of this parabola is located at $(4 - R, 7(5 - R)^2)$. If there are 5 or more Rats, there cannot be enough Cats to counteract the Rats and so the score would be 0. Here is a table showing how the number of Rats impacts the maximum possible score:

Rats	Maximum Possible Score
0	175
1	112
2	63
3	28
4	7
5 or more	0

What is the Probability That Your Turn Will Only Last One More Roll?

We are supposing that you have the following dice showing in the middle of your turn, and that we are keeping the 3.

Cat, Bat, Bat, Bat, Bat, Rat, 3, 5, 5.

What is the probability that re-rolling the two fives will result in either an animal or a 3 on each die?

Each die has a $\frac{4}{6} = \frac{2}{3}$ probability of rolling either an animal or a 3. Since the outcomes of the two dice are independent of one another, we can obtain the overall probability by multiplying the two probabilities. Therefore, the overall probability that the turn will only last one more roll is $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$. That means the probability is slightly greater than one half that your turn will not end with one more roll.

What is the Best Bet?

We are supposing that you have the following dice in the middle of your turn:

Cat, Cat, Cat, Bat, Bat, Rat, 3, 5, 7

Expected Value if You Choose 3 and Stick With That Choice

If the dice showing the 5 and 7 are re-rolled and show either a 5 or 7, it is as if they were not rolled at all since we will not keep those values. Each die has four possibilities for how it could end up. The 16 overall possible outcomes are listed below along with the corresponding total scores. (Remember that Bats count as good animals with 3s.)

- Cat, Cat, Cat, Bat, Bat, Rat, 3, Cat, Cat – Total Score: 21
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Cat, Bat – Total Score: 21
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Cat, Rat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Cat, 3 – Total Score: 36
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Bat, Cat – Total Score: 21
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Bat, Bat – Total Score: 21
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Bat, Rat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Bat, 3 – Total Score: 36
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Rat, Cat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Rat, Bat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Rat, Rat – Total Score: 9
- Cat, Cat, Cat, Bat, Bat, Rat, 3, Rat, 3 – Total Score: 24
- Cat, Cat, Cat, Bat, Bat, Rat, 3, 3, Cat – Total Score: 36

- Cat, Cat, Cat, Bat, Bat, Rat, 3, 3, Bat – Total Score: 36
- Cat, Cat, Cat, Bat, Bat, Rat, 3, 3, Rat – Total Score: 24
- Cat, Cat, Cat, Bat, Bat, Rat, 3, 3, 3 – Total Score: 45

We can condense this list as follows:

Score	Probability	Score × Probability
45	$\frac{1}{16}$	$\frac{45}{16}$
36	$\frac{4}{16}$	$\frac{144}{16}$
24	$\frac{2}{16}$	$\frac{48}{16}$
21	$\frac{4}{16}$	$\frac{84}{16}$
15	$\frac{4}{16}$	$\frac{60}{16}$
9	$\frac{1}{16}$	$\frac{9}{16}$
Expected Value		24.375

We could have obtained the same number by simply taking the average of the 16 possible scores. Either way we calculate this, we see that choosing a 3 results in an expected score of 24.375.

Expected Value if You Choose 5 and Stick With That Choice

If the dice showing the 3 and 7 are re-rolled, each die has four possibilities for how it could end up. The 16 overall possible outcomes are listed below along with the corresponding total scores. (Remember that Bats are neutral with 5s.)

- Cat, Cat, Cat, Bat, Bat, Rat, Cat, 5, Cat – Total Score: 25
- Cat, Cat, Cat, Bat, Bat, Rat, Cat, 5, Bat – Total Score: 20
- Cat, Cat, Cat, Bat, Bat, Rat, Cat, 5, Rat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, Cat, 5, 5 – Total Score: 40
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, 5, Cat – Total Score: 20
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, 5, Bat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, 5, Rat – Total Score: 10
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, 5, 5 – Total Score: 30
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, 5, Cat – Total Score: 15
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, 5, Bat – Total Score: 10
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, 5, Rat – Total Score: 5
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, 5, 5 – Total Score: 20
- Cat, Cat, Cat, Bat, Bat, Rat, 5, 5, Cat – Total Score: 40
- Cat, Cat, Cat, Bat, Bat, Rat, 5, 5, Bat – Total Score: 30
- Cat, Cat, Cat, Bat, Bat, Rat, 5, 5, Rat – Total Score: 20
- Cat, Cat, Cat, Bat, Bat, Rat, 5, 5, 5 – Total Score: 45

We can condense this list as follows:

Score	Probability	Score × Probability
45	$\frac{1}{16}$	$\frac{45}{16}$
40	$\frac{2}{16}$	$\frac{80}{16}$
30	$\frac{2}{16}$	$\frac{60}{16}$
25	$\frac{1}{16}$	$\frac{25}{16}$
20	$\frac{4}{16}$	$\frac{80}{16}$
15	$\frac{3}{16}$	$\frac{45}{16}$
10	$\frac{2}{16}$	$\frac{20}{16}$
5	$\frac{1}{16}$	$\frac{5}{16}$
Expected Value		22.5

We can see that choosing a 5 results in an expected score of 22.5.

Expected Value if You Choose 7 and Stick With That Choice

If the dice showing the 3 and 5 are re-rolled, each die has four possibilities for how it could end up. The 16 overall possible outcomes are listed below along with the corresponding total scores. (Remember that Bats are bad animals like Rats with 7s.)

- Cat, Cat, Cat, Bat, Bat, Rat, Cat, Cat, 7 – Total Score: 21
- Cat, Cat, Cat, Bat, Bat, Rat, Cat, Bat, 7 – Total Score: 7
- Cat, Cat, Cat, Bat, Bat, Rat, Cat, Rat, 7 – Total Score: 7
- Cat, Cat, Cat, Bat, Bat, Rat, Cat, 7, 7 – Total Score: 28
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, Cat, 7 – Total Score: 7
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, Bat, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, Rat, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, Bat, 7, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, Cat, 7 – Total Score: 7
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, Bat, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, Rat, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, Rat, 7, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, 7, Cat, 7 – Total Score: 28
- Cat, Cat, Cat, Bat, Bat, Rat, 7, Bat, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, 7, Rat, 7 – Total Score: 0
- Cat, Cat, Cat, Bat, Bat, Rat, 7, 7, 7 – Total Score: 21

We can condense this list as follows:

Score	Probability	Score × Probability
28	$\frac{2}{16}$	$\frac{56}{16}$
21	$\frac{2}{16}$	$\frac{42}{16}$
7	$\frac{4}{16}$	$\frac{28}{16}$
0	$\frac{8}{16}$	$\frac{0}{16}$
Expected Value		7.875

We can see that choosing a 7 results in an expected score of 7.875.

The Best Option

When you have already rolled the following dice

Cat, Cat, Cat, Bat, Bat, Rat, 3, 5, 7

the best choice is to save the 3 and re-roll the 5 and the 7.

Another Scenario

This time we are supposing that you have the following dice in the middle of your turn:

Cat, Cat, Cat, Cat, Bat, Bat, 3, 5, 7.

If you save the 3 and re-roll the 5 and 7, the 16 possible outcomes are:

- Cat, Cat, Cat, Cat, Bat, Bat, 3, Cat, Cat – Total Score: 27
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Cat, Bat – Total Score: 27
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Cat, Rat – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Cat, 3 – Total Score: 48
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Bat, Cat – Total Score: 27
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Bat, Bat – Total Score: 27
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Bat, Rat – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Bat, 3 – Total Score: 48
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Rat, Cat – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Rat, Bat – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Rat, Rat – Total Score: 15
- Cat, Cat, Cat, Cat, Bat, Bat, 3, Rat, 3 – Total Score: 36
- Cat, Cat, Cat, Cat, Bat, Bat, 3, 3, Cat – Total Score: 48
- Cat, Cat, Cat, Cat, Bat, Bat, 3, 3, Bat – Total Score: 48
- Cat, Cat, Cat, Cat, Bat, Bat, 3, 3, Rat – Total Score: 30
- Cat, Cat, Cat, Cat, Bat, Bat, 3, 3, 3 – Total Score: 63

The expected value of the score if you choose to save the 3 is 33.

If you save the 5 and re-roll the 3 and 7, the 16 possible outcomes are:

- Cat, Cat, Cat, Cat, Bat, Bat, Cat, 5, Cat – Total Score: 35
- Cat, Cat, Cat, Cat, Bat, Bat, Cat, 5, Bat – Total Score: 30

- Cat, Cat, Cat, Cat, Bat, Bat, Cat, 5, Rat – Total Score: 25
- Cat, Cat, Cat, Cat, Bat, Bat, Cat, 5, 5 – Total Score: 60
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, 5, Cat – Total Score: 30
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, 5, Bat – Total Score: 25
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, 5, Rat – Total Score: 20
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, 5, 5 – Total Score: 50
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, 5, Cat – Total Score: 25
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, 5, Bat – Total Score: 20
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, 5, Rat – Total Score: 15
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, 5, 5 – Total Score: 40
- Cat, Cat, Cat, Cat, Bat, Bat, 5, Cat, 5 – Total Score: 60
- Cat, Cat, Cat, Cat, Bat, Bat, 5, Bat, 5 – Total Score: 50
- Cat, Cat, Cat, Cat, Bat, Bat, 5, Rat, 5 – Total Score: 40
- Cat, Cat, Cat, Cat, Bat, Bat, 5, 5, 5 – Total Score: 75

The expected value of the score if you choose to save the 5 is 37.5.

If you save the 7 and re-roll the 3 and 5, the 16 possible outcomes are:

- Cat, Cat, Cat, Cat, Bat, Bat, Cat, Cat, 7 – Total Score: 35
- Cat, Cat, Cat, Cat, Bat, Bat, Cat, Bat, 7 – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, Cat, Rat, 7 – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, Cat, 7, 7 – Total Score: 56
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, Cat, 7 – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, Bat, 7 – Total Score: 7
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, Rat, 7 – Total Score: 7
- Cat, Cat, Cat, Cat, Bat, Bat, Bat, 7, 7 – Total Score: 28
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, Cat, 7 – Total Score: 21
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, Bat, 7 – Total Score: 7
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, Rat, 7 – Total Score: 7
- Cat, Cat, Cat, Cat, Bat, Bat, Rat, 7, 7 – Total Score: 28
- Cat, Cat, Cat, Cat, Bat, Bat, 7, Cat, 7 – Total Score: 56
- Cat, Cat, Cat, Cat, Bat, Bat, 7, Bat, 7 – Total Score: 28
- Cat, Cat, Cat, Cat, Bat, Bat, 7, Rat, 7 – Total Score: 28
- Cat, Cat, Cat, Cat, Bat, Bat, 7, 7, 7 – Total Score: 63

The expected value of the score if you choose to save the 7 is 27.125.

So for this scenario the best option is to save the 5 and re-roll the 3 and the 7.